

Algebraic degrees of stretch factors in mapping class groups

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We explicitly construct pseudo-Anosov maps on the closed surface of genus g with orientable foliations whose stretch factor λ is a Salem number with algebraic degree $2g$. Using this result, we show that there is a pseudo-Anosov map whose stretch factor has algebraic degree d , for each positive even integer d such that $d \leq g$.

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1 Introduction

Let S_g be a closed surface of genus $g \geq 2$. The *mapping class group* of S_g , denoted $\text{Mod}(S_g)$, is the group of isotopy classes of orientation preserving homeomorphisms of S_g . An element $f \in \text{Mod}(S_g)$ is called a *pseudo-Anosov* mapping class if there are transverse measured foliations (\mathcal{F}^u, μ_u) and (\mathcal{F}^s, μ_s) , a number $\lambda(f) > 1$, and a representative homeomorphism ϕ such that

$$\phi(\mathcal{F}^u, \mu_u) = (\mathcal{F}^u, \lambda(f)\mu_u) \text{ and } \phi(\mathcal{F}^s, \mu_s) = (\mathcal{F}^s, \lambda(f)^{-1}\mu_s).$$

In other words, ϕ stretches along one foliation by $\lambda(f)$ and the other by $\lambda(f)^{-1}$. The number $\lambda(f)$ is called the *stretch factor* (or *dilatation*) of f .

A pseudo-Anosov mapping class is said to be orientable if its invariant foliations are orientable. Let $\lambda_H(f)$ be the spectral radius of the action of f on $H_1(S_g; \mathbb{R})$. Then

$$\lambda_H(f) \leq \lambda(f),$$

and the equality holds if and only if the invariant foliations for f are orientable (see [5]). The number $\lambda_H(f)$ is called the *homological stretch factor* of f .

Question Which real numbers can be stretch factors?

It is a long-standing open question. Fried [4] conjectured that $\lambda > 1$ is a stretch factor if and only if all conjugate roots of λ and $1/\lambda$ are strictly greater than $1/\lambda$ and strictly less than λ in magnitude.

Thurston [12] showed that a stretch factor λ is an algebraic integer whose algebraic degree has an upper bound $6g - 6$. More specifically, λ is the largest root in absolute value of a monic palindromic polynomial. Thurston gave a construction of mapping classes of $\text{Mod}(S_g)$ generated by two multitwists and he mentioned that his construction can make a pseudo-Anosov mapping class whose stretch factor has algebraic degree $6g - 6$. However, he did not give specific examples.

What happens if we fix the genus g ? To simplify the question, we may ask which algebraic degrees are possible on S_g .

Question *What degrees of stretch factors can occur on S_g ?*

Very little is known about this question. Using Thurston's construction, it is easy to find quadratic integers as stretch factors. Neuwirth and Patterson [10] found non-quadratic examples, which are algebraic integers of degree 4 and 6 on surfaces of genus 4 and 6, respectively. Using interval exchange maps, Arnoux and Yoccoz [1] gave the first generic construction of pseudo-Anosov maps whose stretch factor has algebraic degree g on S_g for each $g \geq 2$.

Main Theorems

In this paper, we give a generic construction of pseudo-Anosov mapping classes with stretch factor of algebraic degree $2g$.

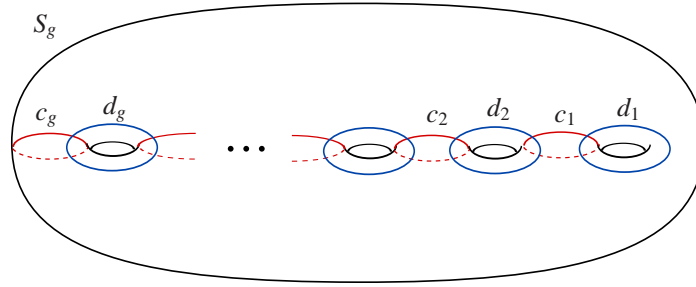
Let c_i and d_j be simple closed curves on S_g as in Figure 1. For $k \geq 3$, let us define

$$f_{g,k} = T_{A_{g,k}} T_{B_g},$$

where $T_{A_{g,k}} = (T_{c_1} T_{c_2} \cdots T_{c_{g-1}}) (T_{c_g})^k$ and $T_{B_g} = T_{d_1} \cdots T_{d_g}$. Here, T_α is the Dehn twist about α . We will show that $f_{g,k}$ is a pseudo-Anosov mapping class and its stretch factor $\lambda(f_{g,k})$ is a special algebraic integer, called Salem number. A *Salem number* is an algebraic integer $\alpha > 1$ whose Galois conjugates other than α have absolute value less than or equal to 1 and at least one conjugate lies on the unit circle.

Theorem A *For each $g \geq 2$ and $k \geq 3$, $f_{g,k}$ is a pseudo-Anosov mapping class and satisfies the following properties:*

- (1) $\lambda(f_{g,k}) = \lambda_H(f_{g,k})$,
- (2) $\lambda(f_{g,k})$ is a Salem number, and
- (3) $\lim_{g \rightarrow \infty} \lambda(f_{g,k}) = k - 1$.

Figure 1: Simple closed curves on S_g

In particular, we will prove that for $k = 4$, the algebraic degree of stretch factor is $2g$. It is known that the degree of the stretch factor of a pseudo-Anosov mapping class $f \in \text{Mod}(S_g)$ with orientable foliations is bounded above by $2g$ (see [12]). Therefore our examples give the maximum degrees of stretch factors for orientable foliations in $\text{Mod}(S_g)$ for each $g \geq 2$.

Theorem B *Let $f_g \in \text{Mod}(S_g)$ be the mapping class given by*

$$f_g = f_{g,4} = T_{A_{g,4}} T_{B_g}.$$

Then the minimal polynomial of the stretch factor $\lambda(f_g)$ is

$$p_g(x) = x^{2g} - 2 \left(\sum_{j=1}^{2g-1} x^j \right) + 1.$$

This implies

$$\deg \lambda(f_g) = 2g.$$

The hard part is to show the irreducibility of $p_g(x)$, which is proved in section 7.

In general, for each $k \geq 3$, the Salem stretch factor of $f_{g,k}$ is the root of the polynomial

$$p_{g,k}(x) = x^{2g} - (k-2) \left(\sum_{j=1}^{2g-1} x^j \right) + 1.$$

It can be shown that $p_{g,k}(x)$ is irreducible for each $k \geq 4$, but since the main purpose of this paper is degree realization, we will prove only for $k = 4$ case that the algebraic degree of the stretch factor is $2g$.

Using a branched cover construction, we use Theorem B to deduce the following partial answer to our question about algebraic degrees.

Corollary 5 *For each positive integer $h \leq g/2$, there is a pseudo-Anosov mapping class $\tilde{f}_h \in \text{Mod}(S_g)$ such that $\deg(\lambda(\tilde{f}_h)) = 2h$ and $\lambda(\tilde{f}_h)$ is a Salem number.*

Obstructions.

There are three known obstructions for the existence of algebraic degrees. For any pseudo-Anosov $f \in \text{Mod}(S_g)$, we have:

- (1) $\deg \lambda(f) \geq 2$,
- (2) $\deg \lambda(f) \leq 6g - 6$, and
- (3) if $\deg \lambda(f) > 3g - 3$, then $\deg \lambda(f)$ is even.

The third obstruction is due to Long [8] and we have another proof in section 5. It turns out these are the only obstructions for $g = 2$. However it is not known whether there are other obstructions of algebraic degrees for $g \geq 3$. By computer search, odd degree stretch factors are rare compared to even degrees. We conjecture that every even degree $d \leq 6g - 6$ can be realized as the algebraic degree of stretch factors.

Conjecture *On S_g , there exists a pseudo-Anosov mapping class with a stretch factor of algebraic degree d for each positive even integer $d \leq 6g - 6$.*

In section 6, we show that the conjecture is true for $g = 2, 3, 4$, and 5.

Outline

In section 2 we will give the basic definitions and results about Thurston's construction. We will prove Theorem A in section 3 by the theory of Coxeter graphs. In section 4, we construct pseudo-Anosov mapping classes via branched covers. In section 5, we explain some properties of odd degree stretch factors. Section 6 contains examples of even degree stretch factors for $g = 2, 3, 4$ and 5. Section 7 is where we prove Theorem B, that is, we prove that the minimal polynomial of $\lambda(f_g)$ has degree $2g$.

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2 Background

Thurston's construction

We recall Thurston's construction of mapping classes [12]. For more details on this material, see [3] or [6].

Suppose $A = \{a_1, \dots, a_n\}$ is a set of pairwise disjoint simple closed curves, called a *multicurve*. We denote the product of Dehn twists $\prod_{i=1}^n T_{a_i}$ by T_A . This product is called a *multitwist*.

Suppose $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_m\}$ are multicurves in a surface S so that $A \cup B$ fills S , that is, the complement of $A \cup B$ is a disjoint union of disks and once-punctured disks. Let N be the $n \times m$ matrix whose (j, k) -entry is the geometric intersection number $i(a_j, b_k)$ of a_j and b_k . Let $\nu = \nu(A \cup B)$ be the largest eigenvalue in magnitude of the matrix NN^t . If $A \cup B$ is connected, then NN^t is primitive and by the Perron–Frobenius theorem ν is a positive real number greater than 1 (see [3, p. 392 - 395] for more detail).

Thurston constructed a singular Euclidean structure on S with respect to which $\langle T_A, T_B \rangle$ acts by affine transformations given by the representation $\rho : \langle T_A, T_B \rangle \rightarrow \text{PSL}(2, \mathbb{R})$

$$\rho(T_A) = \begin{pmatrix} 1 & -\nu^{1/2} \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho(T_B) = \begin{pmatrix} 1 & 0 \\ \nu^{1/2} & 1 \end{pmatrix}.$$

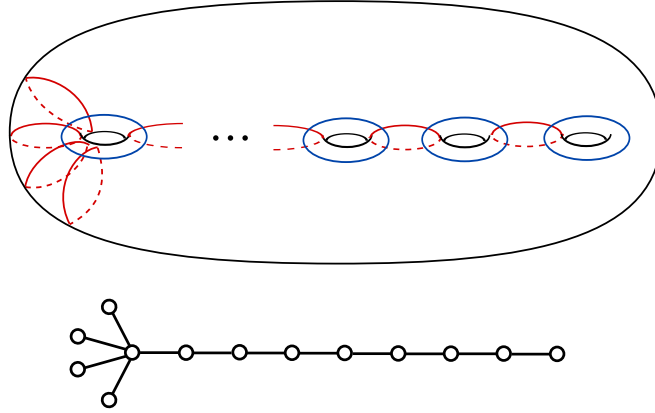
In particular, an element $f \in \langle T_A, T_B \rangle$ is pseudo-Anosov if and only if $\rho(f)$ is a hyperbolic element in $\text{PSL}(2, \mathbb{R})$ and then the stretch factor $\lambda(f)$ is equal to the bigger eigenvalue of $\rho(f)$. For instance, for a mapping class $f = T_A T_B$,

$$\rho(T_A T_B) = \begin{pmatrix} 1 & -\nu^{1/2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \nu^{1/2} & 1 \end{pmatrix} = \begin{pmatrix} 1 - \nu & -\nu^{1/2} \\ \nu^{1/2} & 1 \end{pmatrix},$$

and the stretch factor $\lambda(T_A T_B)$ is the bigger root of the characteristic polynomial

$$\lambda^2 - \lambda(\nu - 2) + 1,$$

provided that $\nu - 2 > 2$.

Figure 2: Multicurves and configuration graph $\mathcal{G}(A_{g,k} \cup B_g)$

3 Proof by the theory of Coxeter graphs

We will prove Theorem A in this section.

For the set C of simple closed curves on the surface S_g , the *configuration graph* for C , denoted $\mathcal{G}(C)$, is the graph with a vertex for each simple closed curve and an edge for every point of intersection between simple closed curves.

Let $f_{g,k}$ be a mapping class on S_g defined by

$$f_{g,k} = T_{A_{g,k}} T_{B_g}, \quad k \geq 3,$$

as in Theorem A. By regarding the multiple power of T_{c_g} as the product of Dehn twists about parallel (isotopic) simple closed curves c_{g_1}, \dots, c_{g_k} , let us define the multicurves

$$A_{g,k} = \{c_1, \dots, c_{g-1}, c_{g_1}, \dots, c_{g_k}\} \text{ and } B_g = \{d_1, \dots, d_g\}.$$

Then the configuration graph $\mathcal{G}(A_{g,k} \cup B_g)$ is a tree as in Figure 2,

3.1 Coxeter graphs and mapping class groups

We say that a finite graph \mathcal{G} is a *Coxeter graph* if there are no self-loops or multiple edges. For given multicurves A and B such that $A \cup B$ fills the surface S , suppose that the configuration graph $\mathcal{G} = \mathcal{G}(A \cup B)$ is a Coxeter graph. Leininger proved the following theorem.

Theorem 1 ([6] Theorem 8.1 and Theorem 8.4) *Let $\mathcal{G}(A \cup B)$ be a non-critical dominant Coxeter graph. Then $T_A T_B$ is a pseudo-Anosov mapping class with stretch factor λ such that*

$$\lambda^2 + \lambda(2 - \mu^2) + 1 = 0,$$

where μ is the spectral radius of the graph \mathcal{G} .

For the definitions and pictures of critical and dominant graphs, see [6, Section 1]

For the multicurves $A_{g,k}$ and B_g in Theorem A, $\mathcal{G}(A_{g,k} \cup B_g)$ is a non-critical dominant Coxeter graph for each $k \geq 3$. Therefore by Theorem 1 the mapping class $f_{g,k} = T_A T_B$ is pseudo-Anosov for each $k \geq 3$.

3.2 Orientability

Suppose that \mathcal{G} is a connected Coxeter graph with the set Σ of vertices. There is an associated quadratic form $\Pi_{\mathcal{G}}$ on \mathbb{R}^{Σ} and a faithful representation

$$\Theta : \mathcal{C}(\mathcal{G}) \rightarrow \mathrm{O}(\Pi_{\mathcal{G}}),$$

where $\mathcal{C}(\mathcal{G})$ is a Coxeter group with generating set Σ , $\mathrm{O}(\Pi_{\mathcal{G}})$ is the orthogonal group of the quadratic form $\Pi_{\mathcal{G}}$, and each generator $s_i \in \Sigma$ is represented by a reflection. Leininger also proved the following theorem.

Theorem 2 ([6] Theorem 8.2) *Let $\mathcal{G}(A \cup B)$ be a Coxeter graph and suppose that A and B can be oriented so that all intersections of A with B are positive. Then there exists a homomorphism*

$$\eta : \mathbb{R}^{\Sigma} \rightarrow H_1(S; \mathbb{R})$$

such that

$$(T_A T_B)_* \circ \eta = -\eta \circ \Theta(\sigma_A \sigma_B),$$

where $\sigma_A \sigma_B$ is an element in $\mathcal{C}(\mathcal{G})$ corresponding to $T_A T_B$.

Moreover, $\Theta(\sigma_A \sigma_B)|_{\ker(\eta)} = -I$ and η preserves spectral radii.

Theorem 2 implies that if A and B can be oriented as in the theorem, then the stretch factor of a pseudo-Anosov mapping class is equal to the spectral radius of the action on homology. For multicurves $A_{g,k}$ and B_g in Theorem A, they can be oriented so that all intersections are positive as in Figure 3. Therefore we have

$$\lambda(f_{g,k}) = \lambda_H(f_{g,k})$$

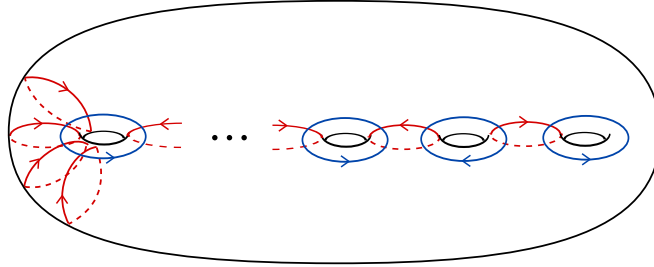
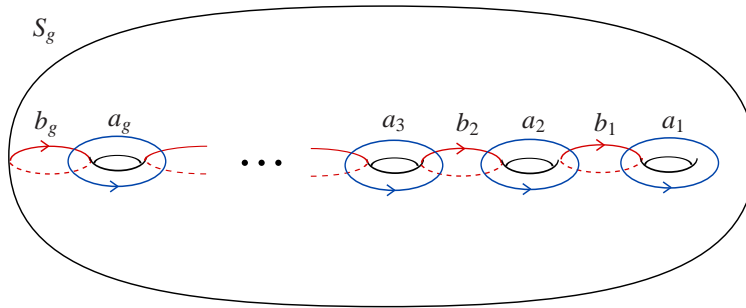


Figure 3: Orientation of positive intersections

and the invariant foliations for $f_{g,k}$ are orientable.

It is also possible to directly compute the action on the first homology. Consider the mapping class $f_g = T_{A_{g,4}}T_{B_g}$ as in Theorem B. Let us choose a basis $\{a_1, b_1, \dots, a_g, b_g\}$ for $H_1(S_g)$ as in Figure 4.

Figure 4: A basis for $H_1(S_g)$.

By computing images of each basis element under f_g , we can get the action on $H_1(S_g)$

$$\begin{pmatrix} 1 & -1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & -1 \\ 4 & 0 & 0 & 0 & \cdots & -3 \end{pmatrix}.$$

By induction, the characteristic polynomial $h_g(x)$ of the homological action is

$$h_g(x) = x^{2g} + 2 \left(\sum_{j=1}^{2g-1} (-1)^j x^j \right) + 1.$$

Since the largest root of $h_g(x)$ in magnitude is a negative real number, we can deduce that the stretch factor $\lambda(f_g)$ is the root of $h_g(-x)$. Specifically, $\lambda(f_g)$ is the root of

$$p_g(x) = x^{2g} - 2 \left(\sum_{j=1}^{2g-1} x^j \right) + 1.$$

In a similar way, one can get the polynomial for $\lambda(f_{g,k})$, which is

$$p_{g,k}(x) = x^{2g} - (k-2) \left(\sum_{j=1}^{2g-1} x^j \right) + 1.$$

3.3 Salem numbers and spectral properties of starlike trees

The configuration graph $\mathcal{G}(A_{g,k} \cup B_g)$ for $f_{g,k}$ is a special type of graphs, called a starlike tree, and its relation to Salem numbers is studied in [9]. A *starlike tree* is a tree with at most one vertex of degree > 2 . Let $T = T(n_1, n_2, \dots, n_k)$ be the starlike tree with k arms of n_1, n_2, \dots, n_k edges.

Theorem 3 ([9] Corollary 9) *Let $T = T(n_1, n_2, \dots, n_k)$ be a starlike tree and let μ be the spectral radius of T . Suppose that μ is not an integer and T is a non-critical dominant graph. Then $\lambda > 1$, defined by $\sqrt{\lambda} + 1/\sqrt{\lambda} = \mu$, is a Salem number.*

The configuration graph $\mathcal{G}(A_{g,k} \cup B_g)$ in Theorem A is a non-critical dominant starlike tree

$$T(2g-2, \underbrace{1, 1, \dots, 1}_{k\text{-times}}), \quad k \geq 3$$

and we will denote it by $T(2g-2, k \cdot 1)$. The fact that the spectral radius of $T(2g-2, k \cdot 1)$ is not an integer follows from the following theorem.

Theorem 4 ([11]) *If μ is the spectral radius of the starlike tree $T(n, k \cdot 1)$, then*

$$\sqrt{k+1} < \mu < \frac{k}{\sqrt{k-1}}$$

for $n \geq 1$ and $k \geq 3$.

Thus for the starlike tree $T(n, k \cdot 1)$, the spectral radius satisfies

$$k + 1 < \mu^2 < \frac{k^2}{k-1} = k + 1 + \frac{1}{k-1}.$$

Therefore μ is not an integer and by Theorem 3 $\lambda(f_{g,k})$ is a Salem number.

Moreover, the proof of Corollary 2.1 of Lepović–Gutman [7] implies that

$$\lim_{g \rightarrow \infty} \lambda(f_{g,k}) = k - 1.$$

For completeness, we reprove this here.

Recall that $\lambda(f_{g,k})$ is the largest root of

$$p_{g,k}(x) = x^{2g} - (k-2) \left(\sum_{j=1}^{2g-1} x^j \right) + 1.$$

By multiplying $p_{g,k}(x)$ by $x-1$, the stretch factor $\lambda(f_{g,k})$ is the largest root in magnitude of

$$q_{g,k}(x) = x^{2g+1} - (k-1)x^{2g} + (k-1)x - 1.$$

We have $q_{g,k}(k-1) = (k-1)^2 - 1 > 0$, and for any fixed positive integer m ,

$$q_{g,k} \left(k-1 - \frac{1}{10^m} \right) = \left(k-1 - \frac{1}{10^m} \right)^{2g} \left(-\frac{1}{10^m} \right) + (k-1) \left(k-1 - \frac{1}{10^m} \right) - 1$$

Hence $q_{g,k} \left(k-1 - 10^{-m} \right) < 0$ for sufficiently large values of g and therefore $p_{g,k}(x)$ has a root on the interval $(k-1 - 10^{-m}, k-1)$. This implies

$$\lim_{g \rightarrow \infty} \lambda(f_{g,k}) = k - 1.$$

This completes the proof of Theorem A.

Remark A positive integer cannot be a stretch factor (which is an algebraic integer of degree 1). However, Theorem A implies that for sufficiently large genus g there is a stretch factor which is a Salem number arbitrarily close to a given integer $k-1$ for each $k \geq 3$.

4 Branched Covers

Lifting a pseudo-Anosov mapping class via a covering map is one way to construct another pseudo-Anosov mapping class. If there is a branched cover $\tilde{S} \rightarrow S$ and a pseudo-Anosov mapping class $f \in \text{Mod}(S)$, then there is some $k \in \mathbb{N}$ such that $\text{Mod}(\tilde{S})$ has a pseudo-Anosov element \tilde{f} which is a lift of f^k and hence $\lambda(\tilde{f}) = \lambda(f)^k$.

Corollary 5 *Let $g \geq 2$. For each positive integer $h \leq g/2$, there is a pseudo-Anosov mapping class $\tilde{f}_h \in \text{Mod}(S_g)$ such that $\deg(\lambda(\tilde{f}_h)) = 2h$ and $\lambda(\tilde{f}_h)$ is a Salem number.*

Proof Let

$$h = \begin{cases} \frac{g-2m}{2}, & \text{if } g \text{ is even, } m = 0, 1, \dots, (g-2)/2, \\ \frac{g-1-2m}{2}, & \text{if } g \text{ is odd, } m = 0, 1, \dots, (g-3)/2. \end{cases}$$

Then h is an integer such that $1 \leq h \leq g/2$.

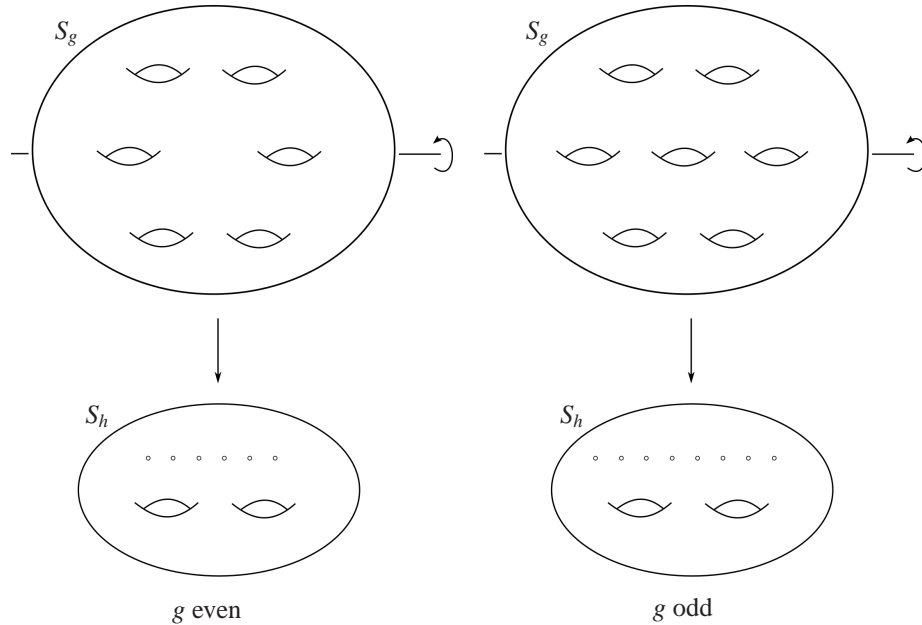


Figure 5: A branched cover

Construct a branched cover $S_g \rightarrow S_h$ as in Figure 5. For $h \geq 2$, S_h has a pseudo-Anosov mapping class $f_h \in \text{Mod}(S_h)$ as in the Theorem B whose stretch factor has

$\deg(\lambda(f_h)) = 2h$. For some k , f_h^k lifts to S_g and the lift has stretch factor $\lambda(f_h)^k$. We claim that $\deg(\lambda(f_h)^k) = 2h$. To see this, let λ_i , $1 \leq i \leq 2h$, be the roots of the minimal polynomial of $\lambda(f_h)$ and let us define a polynomial

$$p(x) = \prod_{i=1}^{2h} (x - \lambda_i^k).$$

Then $p(x)$ is an integral polynomial because the following elementary symmetric polynomials in $\lambda_1, \dots, \lambda_{2h}$

$$\sum \lambda_i, \sum_{i < j} \lambda_i \lambda_j, \sum_{i < j < l} \lambda_i \lambda_j \lambda_l, \dots, \lambda_1 \lambda_2 \cdots \lambda_{2h}$$

are all integers and hence each coefficient of $p(x)$

$$\sum \lambda_i^k, \sum_{i < j} \lambda_i^k \lambda_j^k, \sum_{i < j < l} \lambda_i^k \lambda_j^k \lambda_l^k, \dots, \lambda_1^k \lambda_2^k \cdots \lambda_{2h}^k$$

are integers as well. Therefore $p(x)$ is divided by the minimal polynomial of $\lambda(f_h)^k$. Due to the proof of Theorem B in section 7, $\lambda(f_h)^k$ is also a Salem number and $p(x)$ does not have a cyclotomic factor. This implies that $p(x)$ is irreducible and $\deg(\lambda(f_h)^k) = 2h$.

If $h = 1$, S_h is a torus and it admits a Anosov mapping class f whose stretch factor $\lambda(f)$ has algebraic degree 2. Then similar arguments as above tells us that there is a lift of some power of f to S_g whose stretch factor has $\deg(\lambda(f^k)) = 2$.

Therefore there is a pseudo-Anosov map $\tilde{f}_h \in \text{Mod}(S_g)$ with $\deg(\lambda(\tilde{f}_h)) = 2h$ for each $h \leq g/2$. In other words, every positive even degree $d \leq g$ is realized as the algebraic degree of a stretch factor on S_g . \square

5 Stretch factors of odd degrees

Long proved the following degree obstruction and McMullen communicated to us the following proof. First we will give a definition of the reciprocal polynomial. Given a polynomial $p(x)$ of degree d , we define the reciprocal polynomial $p^*(x)$ of $p(x)$ by $p^*(x) = x^d p(1/x)$. It is a well-known property that $p^*(x)$ is irreducible if and only if $p(x)$ is irreducible.

Theorem 6 ([8]) *Let $f \in \text{Mod}(S_g)$ be a pseudo-Anosov mapping class having stretch factor $\lambda(f)$. If $\deg(\lambda(f)) > 3g - 3$, then $\deg(\lambda(f))$ is even.*

Proof Since f acts by a piecewise integral projective transformation on the $6g - 6$ dimensional space \mathcal{PMF} of projective measured foliations on S_g , and since $\lambda(f)$ is an eigenvalue of this action, $\lambda(f)$ is an algebraic integer with $\deg(\lambda(f)) \leq 6g - 6$. Also, since f preserves the symplectic structure on \mathcal{PMF} , it follows that $\lambda(f)$ is the root of palindromic polynomial $p(x)$ whose degree is bounded above by $6g - 6$.

Let $q(x)$ be the minimal polynomial of $\lambda(f)$ and let $q^*(x)$ be the reciprocal polynomial of $q(x)$. Then either $q(x) = q^*(x)$ or they have no common roots, because if there is at least one common root ζ of $q(x)$ and $q^*(x)$, then both $q(x)$ and $q^*(x)$ are the minimal polynomial of ζ and hence $q(x) = q^*(x)$. Suppose $\deg(q(x)) > 3g - 3$. If $q(x)$ and $q^*(x)$ have no common roots, then their product $q(x)q^*(x)$ is a factor of $p(x)$ since $q^*(x)$ is the minimal polynomial of $1/\lambda(f)$. This is a contradiction because $\deg(p(x)) \leq 6g - 6$ but $\deg(q(x)q^*(x)) > 6g - 6$. Therefore we must have $q(x) = q^*(x)$ and this implies that $q(x)$ is an irreducible palindromic polynomial. Hence $\deg(q(x))$ is even since roots of $q(x)$ come in pairs, λ_i and $1/\lambda_i$. \square

It follows from the previous proof that if the minimal polynomial $p(x)$ of λ has odd degree, then $p(x)$ is not palindromic and in fact the minimal palindromic polynomial containing λ as a root is $p(x)p^*(x)$.

We will now show that the stretch factors of degree 3 have an additional special property. A *Pisot number*, also called a *Pisot–Vijayaraghavan number* or a *PV number*, is an algebraic integer greater than 1 such that all its Galois conjugates are strictly less than 1 in absolute value.

Proposition 7 *Let $f \in \text{Mod}(S_g)$. If $\deg(\lambda(f)) = 3$, then $\lambda(f)$ is a Pisot number.*

Proof Let $\lambda_1 > 1$ be the stretch factor of a pseudo-Anosov mapping class with algebraic degree 3, and let $p(x)$ be the minimal polynomial of λ_1 . Let λ_1, λ_2 , and λ_3 be the roots of $p(x)$. Then the degree of $p(x)p^*(x)$ is 6 and it has pairs of roots $(\lambda_1, 1/\lambda_1), (\lambda_2, 1/\lambda_2), (\lambda_3, 1/\lambda_3)$, where λ_1 is the largest root in absolute value. We claim that the absolute values of λ_2 and λ_3 are strictly less than 1.

Suppose one of them has absolute value greater than or equal to 1, say $|\lambda_2| \geq 1$. The constant term $\lambda_1\lambda_2\lambda_3$ of $p(x)$ is ± 1 since it is the factor of a palindromic polynomial with constant term 1. Hence $|\lambda_1\lambda_2\lambda_3| = 1$ and we have

$$\frac{1}{|\lambda_3|} = |\lambda_1\lambda_2| \geq |\lambda_1|,$$

which is a contradiction to the fact that the stretch factor λ_1 is strictly greater than all other roots of the palindromic polynomial $p(x)p^*(x)$. This proves the claim and hence the stretch factor of degree 3 is a Pisot number. \square

We now explain two constructions of mapping classes $f \in \text{Mod}(S_g)$ whose degree of $\lambda(f)$ is odd.

1. As we mentioned, Arnoux–Yoccoz [1] gave examples of a pseudo-Anosov mapping class on S_g whose stretch factor has algebraic degree g . In particular for odd g , this gives examples of mapping classes with odd degree stretch factors. They proved that these stretch factors are all Pisot numbers.
2. For genus 2, there is a pseudo-Anosov mapping class f whose stretch factor has algebraic degree 3 (see section 6). This is the only possible odd degree on S_2 by Long’s obstruction. It is also true that $\deg(\lambda(f)^k) = 3$ for each k because the stretch factor is a Pisot number (Proposition 7). There is a cover $S_g \rightarrow S_2$ for each g , so the lift of some power of f has a stretch factor with algebraic degree 3 on S_g .

Proposition 8 *For each genus g , the stretch factor with algebraic degree 3 can occur on S_g .*

Question *Are there stretch factors with odd algebraic degree that are not Pisot numbers?*

6 Examples of even degrees

Tables 1 through 4 give explicit examples of pseudo-Anosov mapping classes whose stretch factors realize various degrees. We will follow the notation of the software `Xtrain` by Brinkmann. More specifically, a_i, b_i, c_i , and d_i are Dehn twists along standard curves and A_i, B_i, C_i , and D_i are the inverse twists as in [2]. The only missing degree on S_3 is degree 5. We do not know if there is a degree 5 example or there is another degree obstruction.

deg	$f \in \text{Mod}(S_2)$	Minimal polynomial	$\lambda(f)$
2	$a_0 a_0 d_0 C_0 D_1 C_0$	$x^2 - 3x + 1$	$\lambda = 2.618$
3	$a_0 d_0 d_0 C_0 C_0 D_1$	$x^3 - 3x^2 - x - 1$	$\lambda = 3.383$
4	$a_0 d_0 d_0 d_1 c_0 d_0$	$x^4 - x^3 - x^2 - x + 1$	$\lambda = 1.722$
6	$a_0 a_0 d_0 A_0 C_0 D_1$	$x^6 - x^5 - 4x^3 - x + 1$	$\lambda = 2.015$

Table 1: Examples of genus 2

deg	$f \in \text{Mod}(S_3)$	Minimal polynomial	$\lambda(f)$
2	$a_1c_0d_0c_0d_2C_1D_1$	$x^2 - 4x + 1$	3.732
3	$a_0c_0d_0C_1D_1D_2$	$x^3 - 2x^2 + x - 1$	1.755
4	$a_1c_0d_0a_1c_1d_1d_2$	$x^4 - x^3 - 2x^2 - x + 1$	1.722
6	$a_0c_0d_0d_2C_1D_1$	$x^6 - 3x^5 + 3x^4 - 7x^3 + 3x^2 - 3x + 1$	2.739
8	$a_0c_0d_0d_1C_1D_2$	$x^8 - x^7 - 2x^5 - 2x^3 - x + 1$	1.809
10	$a_1c_0d_0d_1C_1A_2D_2$	$x^{10} - x^9 - 2x^8 + 2x^7 - 2x^5 + 2x^3 - 2x^2 - x + 1$	1.697
12	$a_1c_1c_0d_1d_2A_0D_0$	$x^{12} - x^{11} - x^9 - x^8 + x^7 + x^5 - x^4 - x^3 - x + 1$	1.533

Table 2: Examples of genus 3

deg	$f \in \text{Mod}(S_4)$	deg	$f \in \text{Mod}(S_4)$
4	$a_0a_0a_1c_0d_0c_1d_1c_2d_2c_3d_3$	12	$a_0B_1d_0c_0d_1c_1d_2c_2d_3c_3$
6	$a_0B_2A_3d_0c_0d_1c_1d_2c_2d_3c_3$	14	$a_0d_0B_0d_0c_0d_1c_1d_2c_2d_3c_3$
8	$a_0A_1d_0c_0d_1c_1d_2c_2d_3c_3$	16	$A_0d_0c_0d_1c_1d_2c_2d_3c_3$
10	$a_0b_1A_2d_0c_0d_1c_1d_2c_2d_3c_3$	18	$a_0B_1A_2d_0c_0d_1c_1d_2c_2d_3c_3$

Table 3: Examples of genus 4

deg	$f \in \text{Mod}(S_5)$	deg	$f \in \text{Mod}(S_5)$
6	$b_3d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$	16	$a_1B_2d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$
8	$a_0a_1d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$	18	$a_1B_0d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$
10	$a_1A_4d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$	20	$a_1A_0d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$
12	$b_2C_2d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$	22	$a_2A_1d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$
14	$a_1B_1d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$	24	$c_2A_2d_0c_0d_1c_1d_2c_2d_3c_3d_4c_4$

Table 4: Examples of genus 5

7 Irreducibility of Polynomials

In this section, we will prove Theorem B. It is enough to show that the polynomial

$$p_n(x) = x^{2n} - 2 \left(\sum_{j=1}^{2n-1} x^j \right) + 1.$$

is irreducible for $n \geq 2$. We will show that $p_n(x)$ does not have a cyclotomic polynomial factor. It then follows from Kronecker's theorem that $p_n(x)$ is irreducible.

Suppose $p_n(x)$ has the m th cyclotomic polynomial factor for some $m \in \mathbb{N}$. Then $e^{2\pi i/m}$ is a root of $p_n(x)$. Multiplying $p_n(x)$ by $x - 1$ yields

$$x^{2n+1} - 3x^{2n} + 3x - 1$$

and hence we have

$$(1) \quad e^{2(2n+1)\pi i/m} - 3e^{4n\pi i/m} + 3e^{2\pi i/m} - 1 = 0.$$

Consider the real part and the complex part of (1). Then we have the system of equations

$$\begin{cases} \cos\left(\frac{2(2n+1)\pi}{m}\right) - 3\cos\left(\frac{4n\pi}{m}\right) + 3\cos\left(\frac{2\pi}{m}\right) - 1 = 0 \\ \sin\left(\frac{2(2n+1)\pi}{m}\right) - 3\sin\left(\frac{4n\pi}{m}\right) + 3\sin\left(\frac{2\pi}{m}\right) = 0 \end{cases}$$

Using double-angle formula for the first cosine and sum-to-product formula for the last two cosines, the first equation gives

$$2\sin\left(\frac{(2n+1)\pi}{m}\right) \left[3\sin\left(\frac{(2n-1)\pi}{m}\right) - \sin\left(\frac{(2n+1)\pi}{m}\right) \right] = 0.$$

Similarly the second equation gives

$$2\cos\left(\frac{(2n+1)\pi}{m}\right) \left[\sin\left(\frac{(2n+1)\pi}{m}\right) - 3\sin\left(\frac{(2n-1)\pi}{m}\right) \right] = 0.$$

Since sine and cosine have no common zeros, we must have

$$\sin\left(\frac{(2n+1)\pi}{m}\right) - 3\sin\left(\frac{(2n-1)\pi}{m}\right) = 0.$$

For $m \leq 5$, by direct calculation we can see that $p_n(e^{2\pi i/m}) \neq 0$. So we may assume that $m \geq 6$. Let $\varphi = (2n-1)\pi/m$ and then we can write the above equation as

$$(2) \quad \sin\left(\varphi + \frac{2\pi}{m}\right) - 3\sin(\varphi) = 0.$$

Since $\sin(\varphi + 2\pi/m)$ is a real number between -1 and 1 , we have

$$(3) \quad -\frac{1}{3} \leq \sin(\varphi) \leq \frac{1}{3}.$$

Let $\psi = \sin^{-1}(1/3)$. Then note that $\psi < \pi/6$. Equation (3) gives the restriction on φ , which is

$$-\psi \leq \varphi \leq \psi \text{ or } \pi - \psi \leq \varphi \leq \pi + \psi.$$

Another observation from (2) is that both $\sin(\varphi + 2\pi/m)$ and $\sin(\varphi)$ must have the same sign.

We claim that φ has to be on the either first or third quadrant. Suppose φ is on the second quadrant, that is, $\pi - \psi < \varphi < \pi$. Note that $m \geq 6$ implies $2\pi/m \leq \pi/3$.

Since φ is above the x -axis, $\varphi + 2\pi/m$ also has to be above the x -axis due to (2) and hence the only possibility is that $\varphi + 2\pi/m$ is between φ and π . Then

$$0 < \sin\left(\varphi + \frac{2\pi}{m}\right) < \sin(\varphi) \implies \sin\left(\varphi + \frac{2\pi}{m}\right) < 3\sin(\varphi),$$

which is a contradiction to (2). Similar arguments hold if φ is on the fourth quadrant. Therefore the possible range for φ is

$$0 < \varphi \leq \psi \text{ or } \pi < \varphi \leq \pi + \psi.$$

Suppose φ is on the first quadrant. Then so is $\varphi + 2\pi/m$ because

$$0 < \varphi + \frac{2\pi}{m} \leq \psi + \frac{\pi}{3} < \frac{\pi}{2}.$$

We can write

$$\varphi = \frac{(2n-1)\pi}{m} \equiv \frac{j\pi}{m} \pmod{2\pi}$$

for some positive integer j , i.e., $0 < j\pi/m < \pi/2$.

If $j \geq 2$, Using the subadditivity of $\sin(x)$ on the first quadrant

$$\sin(x+y) \leq \sin(x) + \sin(y),$$

we have

$$\begin{aligned} \sin\left(\varphi + \frac{2\pi}{m}\right) - 3\sin(\varphi) &\leq \left(\sin(\varphi) + \sin\left(\frac{2\pi}{m}\right)\right) - 3\sin(\varphi) \\ &= \sin\left(\frac{2\pi}{m}\right) - 2\sin(\varphi) \\ &= \sin\left(\frac{2\pi}{m}\right) - 2\sin\left(\frac{j\pi}{m}\right) < 0, \end{aligned}$$

which contradicts (2).

If $j = 1$, using triple-angle formula

$$\begin{aligned} \sin\left(\varphi + \frac{2\pi}{m}\right) - 3\sin(\varphi) &= \sin\left(\frac{3\pi}{m}\right) - 3\sin\left(\frac{\pi}{m}\right) \\ &= \left(3\sin\left(\frac{\pi}{m}\right) - 4\sin^3\left(\frac{\pi}{m}\right)\right) - 3\sin\left(\frac{\pi}{m}\right) \\ &= -4\sin^3\left(\frac{\pi}{m}\right) < 0, \end{aligned}$$

which contradicts (2) again. Therefore there is no possible φ on the first quadrant. By using the same arguments, the fact that φ is on the third quadrant gives a contradiction. Therefore we can conclude that $p(x)$ does not have a cyclotomic factor.

We now show that $p_n(x)$ is irreducible over \mathbb{Z} . Suppose $p_n(x)$ is reducible and write $p_n(x) = g(x)h(x)$ with non-constant functions $g(x)$ and $h(x)$. There is only one root of $p_n(x)$ whose absolute value is strictly greater than 1. Therefore one of $g(x)$ or $h(x)$ has all roots inside the unit disk. By Kronecker's theorem, this polynomial has to be a product of cyclotomic polynomials, which is a contradiction because $p_n(x)$ does not have a cyclotomic polynomial factor. Therefore $p_n(x)$ is irreducible.

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